Closed-form weak-field expansion of two-loop Euler-Heisenberg Lagrangians

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Abstract

We obtain closed-form expressions, in terms of the Faulhaber numbers, for the weak-field expansion coefficients of the two-loop Euler-Heisenberg effective Lagrangians in a magnetic or electric field. This follows from the observation that the magnetic worldline Green's function has a natural expansion in terms of the Faulhaber numbers.

1 Introduction: The Euler-Heisenberg Lagrangian at one and two loops

The Euler-Heisenberg Lagrangian [1, 2] describes the effect of a virtual electron - positron pair on an external Maxwell field in the one loop and constant field approximation. Its standard proper time representation is

$$\begin{aligned}\mathcal{L}^{(1)}_{\rm spin}(F) &= -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\biggl[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} \\ &\qquad -\frac{1}{3}(a^2-b^2)T^2-1\biggr]. \end{aligned} \tag{1.1}$$

Here T is the proper-time of the loop fermion, m its mass, and a, b are the two Maxwell invariants, related to $\mathbf{E}$, $\mathbf{B}$ by $a^2 - b^2 = B^2 - E^2, \quad ab = \mathbf{E}\cdot\mathbf{B}$. The superscript "(1)" stands for one loop. A similar representation exists for scalar QED [2, 3]:

$$\begin{aligned}\mathcal{L}^{(1)}_{\rm scal}(F) &= \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\biggl[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} \\ &\qquad +\frac{1}{6}(a^2-b^2)T^2-1\biggr]. \end{aligned} \tag{1.2}$$

The Lagrangians (1.1), (1.2) historically provided the first examples for the concept of an effective Lagrangian, and moreover the first nonperturbative result in quantum field theory. See [4] for a review of their many applications and generalizations.

The proper time integrals in these formulas can be done exactly in terms of certain special functions [4]. Alternatively, one can expand the integrands as power series in the field invariants, using the Taylor expansions

$$\frac{z}{\tanh(z)} = \sum_{n=0}^\infty \frac{\mathcal{B}_{2n}}{(2n)!}(2z)^{2n}, \tag{1.3}$$

$$\frac{z}{\sinh(z)} = -\sum_{n=0}^\infty\Bigl(1-2^{1-2n}\Bigr)\frac{\mathcal{B}_{2n}}{(2n)!}(2z)^{2n}. \tag{1.4}$$

Here the $\mathcal{B}_{2n}$ are the Bernoulli numbers. The terms in this expansion involving $N = 2n$ powers of the field contain the information on the low energy

limits of the N photon scattering amplitudes. Thus in the low energy limit one can obtain these amplitudes in closed form [5].
For a purely magnetic field eqs. (1.1), (1.2), (1.3), (1.4) yield

$$\mathcal{L}^{(1)}_{\rm spin}(B) = -8\left(\frac{\alpha}{\pi}\right)B^2\sum_{n=0}^{\infty}\frac{2^{2n}\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{eB}{m^2}\right)^{2n+2}, \quad (1.5)$$

$$\mathcal{L}^{(1)}_{\rm scal}(B) = 4\left(\frac{\alpha}{\pi}\right)B^2\sum_{n=0}^{\infty}\frac{2^{2n}(2^{-2n-3}-1)\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{eB}{m^2}\right)^{2n+2}. \quad (1.6)$$

An analysis of the coefficients shows that these series are divergent but Borel summable [6, 7, 8]. Moreover, the summability relates to the fact that the effective Lagrangian is real in the magnetic case. The corresponding series for the purely electric case is obtained by the replacement $B^2 \to -E^2$, which turns the alternating series into a non-alternating one. The non-alternating series is not Borel summable, which is an indication of the well-known fact that the electric effective Lagrangian has an imaginary part. Although this imaginary part is nonperturbative in nature, it is possible to calculate it from the expansion coefficients by an analysis of their large n behaviour, combined with a Borel dispersion relation [6, 7, 8].

This paper addresses the possibility of analyzing such weak-field expansions at higher loop orders in QED. The two loop corrections to the Lagrangians (1.1), (1.2) have first been considered by Ritus, who obtained them in terms of integral representations both in spinor [9, 10] and scalar QED [11]. Other representations for the same Lagrangians were later given in [12, 13, 14, 15], however they all involve the same apparently intractable type of double integrals. As a consequence, at the two loop level presently only the first few coefficients of the power series expansions for $\mathcal{L}^{(2)}_{\rm spin/scal}(B)$ are known [8], and the imaginary parts ${\rm Im}\mathcal{L}^{(2)}_{\rm spin/scal}(E)$ are known explicitly only in the leading weak field limit [16, 8] (although Lebedev and Ritus succeeded in establishing the general structure of the imaginary part [16]).

This should be contrasted with the case of a self-dual (euclidean) field, corresponding to external photon lines of definite helicity, where the two-loop effective Lagrangians can be obtained in closed form [17, 18, 19]. This result allows one to extend the aforementioned calculation of the low energy limit of the N photon amplitudes to the two loop level for one particular component of the photon S matrix, the so-called MHV amplitudes [18].

For general fields, it seems difficult to make further progress at the two loop level without having a closed form expression for the weak field expansion coefficients. It is the purpose of the present paper to derive such expressions for the purely magnetic (or purely electric) cases. Thus, our goal is to obtain the coefficients of the weak field expansion of the (renormalized) magnetic two-loop effective Euler-Heisenberg Lagrangians $\mathcal{L}^{(2)}_{\rm spin/scal}(B)$,

$$\mathcal{L}^{(2)}_{\rm spin}(B) = \Big(\frac{\alpha}{4\pi}\Big)^2 B^2 \sum_{n=0}^{\infty} a^{(2)}_{\rm spin}(n)\Big(\frac{eB}{m^2}\Big)^{2n+2}, \tag{1.7}$$

$$\mathcal{L}^{(2)}_{\rm scal}(B) = -\frac{1}{2}\Big(\frac{\alpha}{4\pi}\Big)^2 B^2 \sum_{n=0}^{\infty} a^{(2)}_{\rm scal}(n)\Big(\frac{eB}{m^2}\Big)^{2n+2}. \tag{1.8}$$

Note the difference in normalization between the scalar and spinor QED cases, which takes into account the global factor of -2 from statistics and degrees of freedom.

2 Two loop expansion coefficients for spinor QED

We start with the following integral representation of the on-shell renormalized Lagrangian, which was obtained in [14]:

$$\mathcal{L}^{(2)}_{\rm spin}(B) = \mathcal{L}^{(2)}_{\rm spin,main}(B) + \mathcal{L}^{(2)}_{\rm spin,\delta m}(B). \tag{2.1}$$

Here the "main part" is given by the following two parameter integral,

$$\mathcal{L}^{(2)}_{\rm spin,main}(B) = \Big(\frac{\alpha}{4\pi}\Big)^2 B^2 \int_0^\infty \frac{dz}{z^3} e^{-\frac{m^2}{eB}z} \int_0^1 du\,\Big[L(z,u) - L_{02}(z,u) - \frac{g(z)}{G}\Big] \tag{2.2}$$

where $z = eBT$ and

$$
\begin{aligned}
L(z,u) &= \frac{z}{\tanh(z)}\left\{ B_1 \frac{\ln(G/G^z)}{(G-G^z)^2} + \frac{B_2}{G^z(G-G^z)} + \frac{B_3}{G(G-G^z)} \right\},\\
B_1 &= 4z\Big(\coth(z) - \tanh(z)\Big)G^z - 4G\,,\\
B_2 &= 2\dot{G}\dot{G}^z + z(8\tanh(z) - 4\coth(z))G^z - 2\,,\\
B_3 &= 4G - 2\dot{G}\dot{G}^z - 4z\tanh(z)G^z + 2\,,\\
L_{02}(z,u) &= -\frac{12}{G} + 2z^2\,,\\
g(z) &= -6\left[\frac{z^2}{\sinh(z)^2} + z\coth(z) - 2\right]\,.
\end{aligned}
\tag{2.3}
$$

The integrand involves the so-called worldline Green's function $G(u)$ and the "magnetic" Green's function $G^z(z,u)$, as well as their u-derivatives (denoted as $\dot{G}$ and $\dot{G}^z$, respectively):

$$
\begin{aligned}
G &= u(1-u)\,,\\
\dot{G} &= 1 - 2u\,,\\
G^z &= \frac{1}{2}\frac{\cosh(z) - \cosh(z\dot{G})}{z\sinh(z)}\,,\\
\dot{G}^z &= \frac{\sinh(z\dot{G})}{\sinh(z)}\,.
\end{aligned}
\tag{2.4}
$$

The function $L(z,u)$ is essentially the two-loop integrand before renormalization. The term $L_{02}(z,u)$ removes the order z^0, z^2 terms, which implements the renormalization of the charge and the field, and the removal of the vacuum energy. (In the following we will not always make this subtraction explicit.) The other subtraction term involving $\frac{g(z)}{G}$ relates to mass renormalization, which is necessary starting at the two loop level. In the worldline formalism the need for this mass renormalization subtraction can be recognized from the appearance of singularities at $u = 0$ and $u = 1$ [13, 14]. Those singular terms can be absorbed into a mass shift of the one-loop lagrangian, $\delta m_0 \frac{\partial}{\partial m_0}\mathcal{L}^{(1)}_{\text{spin}}(B)$, however this leaves a finite remainder $\mathcal{L}^{(2)}_{\text{spin},\delta\text{m}}(B)$,

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\text{spin},\delta\text{m}}(B) &= -\frac{\alpha}{(4\pi)^3} eBm^2 \int_0^\infty \frac{dz}{z^2}\, \mathrm{e}^{-\frac{m^2}{eB}z}\left[\frac{z}{\tanh(z)} - \frac{z^2}{3} - 1\right]\\
&\qquad \times\left[18 - 12\gamma - 12\ln\left(\frac{m^2 z}{eB}\right) + 12\frac{eB}{m^2 z}\right].
\end{aligned}
\tag{2.5}
$$

The weak field expansion coefficients of $\mathcal{L}^{(2)}_{\mathrm{spin},\delta\mathrm{m}}(B)$ are again easy to obtain using the Taylor expansion (1.3). One finds

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\mathrm{spin},\delta\mathrm{m}}(B) &= \Bigl(\frac{\alpha}{4\pi}\Bigr)^2 B^2 \sum_{n=0}^{\infty} a^{(2)}_{\mathrm{spin},\delta\mathrm{m}}(n)\Bigl(\frac{eB}{m^2}\Bigr)^{2n+2}, \\
a^{(2)}_{\mathrm{spin},\delta\mathrm{m}}(n) &= -12\frac{2^{2n+4}\mathcal{B}_{2n+4}}{(2n+4)(2n+3)}\Bigl(\frac{3}{2}-\gamma-\psi(2n+2)\Bigr). \qquad (2.6)
\end{aligned}
$$

However, the main integral term (2.2) is much more difficult to expand. A brute-force expansion was done in [8], but no closed-form expression was obtained. The technical challenge is to expand the integrand (excluding the $e^{-\frac{m^2}{eB}z}$ factor) in a series in z, in such a way that the u integrals can also be done easily. This is complicated by the fact that the magnetic Green's function $G^z(z,u)$ couples these two parametric variables in a nontrivial manner. The main observation of this paper is that there exists an expansion of $G^z(z,u)$ which decouples the variables in an elegant manner. Remarkably, $G^z(z,u)$ is essentially the generating function of the so-called Faulhaber numbers of number theory [20]. Namely,

$$
\begin{aligned}
G^z(z,u) &= \frac{1}{2}\frac{\cosh(z)-\cosh(z\dot{G})}{z\sinh(z)} \\
&= G(u) - \sum_{m=1}^{\infty}(2z)^{2m}\sum_{k=1}^{m}\tilde{f}(m,k)\,[G(u)]^{k+1} \quad . \qquad (2.7)
\end{aligned}
$$

Here we have defined

$$
\tilde{f}(m,k) \equiv \frac{(-1)^{k+1}}{(2m+1)!}f(m,k) \qquad (2.8)
$$

with $f(m,k)$ the Faulhaber numbers. These numbers can in turn be written as linear combinations of the Bernoulli numbers [20]:

$$
f(m,k) = (-1)^{k+1}\sum_{j=0}^{\lfloor (k-1)/2\rfloor}\frac{1}{k-j}\binom{2k-2j}{k+1}\binom{2m+1}{2j+1}\mathcal{B}_{2m-2j} \qquad (2.9)
$$

$(m,k\geq 1)$.

Given such an expansion, the u integrals are now trivial as they involve powers of the free Green's function $G(u) = u(1-u)$. Using the standard Euler beta function integral we define

$$\int_0^1 du\,\bigl[u(1-u)\bigr]^n \;=\; \frac{n!^2}{(2n+1)!} \equiv \beta(n+1). \tag{2.10}$$

To illustrate the Faulhaber expansion (2.7), consider the first few terms:

$$G^z = G + \Bigl(1 - z\coth(z)\Bigr)G^2 + O(G^3) = G - \frac{1}{3}G^2 z^2 + O(z^4, G^2)\,. \tag{2.11}$$

This suggests rearranging $L(z,u)$ as a series in $\Delta G/G$ where $\Delta G = G^z - G$ is the difference between the magnetic and free worldline Green's functions. After simple manipulations, this leads to

$$\begin{aligned} L(z,u) \;&=\; \frac{4z^2}{G} + \sum_{i=0}^{\infty}\Bigl(-\frac{\Delta G}{G}\Bigr)^i \Bigl\{-\frac{z^2}{\sinh(z)^2}\frac{4}{G}\frac{1}{(i+1)(i+2)} \\ &\quad -\frac{z}{\tanh(z)}\frac{1}{G}\left(8 + \frac{4}{i+2}\right) + \frac{z}{\tanh(z)}\frac{2\,\dot G(\dot G_z - \dot G)}{G^2}\Bigr\}. \end{aligned} \tag{2.12}$$

The u derivatives appearing in the last term can be removed by an integration by parts, yielding

$$\begin{aligned} L(z,u) \;&=\; \frac{4z^2}{G} + 4\frac{z}{\tanh(z)}\Bigl(\frac{z}{\tanh(z)} - 1\Bigr) \\ &+ \sum_{i=0}^{\infty}\Bigl(-\frac{\Delta G}{G}\Bigr)^i \Bigl\{-\frac{z^2}{\sinh(z)^2}\frac{4}{G}\frac{1}{(i+1)(i+2)} - \frac{z}{\tanh(z)}\frac{1}{G}\left(8+\frac{4}{i+2}\right)\Bigr\} \\ &-\frac{2z}{\tanh(z)}\sum_{i=1}^{\infty}\Bigl(-\frac{\Delta G}{G}\Bigr)^i \frac{1}{i}\left\{\frac{2-4(i+1)}{G} + \frac{i+1}{G^2}\right\} \quad . \end{aligned} \tag{2.13}$$

Performing the subtraction, implied in (2.2), of all terms $O(\frac{1}{G})$, we finally obtain

$$\mathcal{L}^{(2)}_{\rm spin,main}(B) \;=\; \Bigl(\frac{\alpha}{4\pi}\Bigr)^2 B^2 \int_0^\infty \frac{dz}{z^3}\,\mathrm{e}^{-\frac{m^2}{eB}z}\int_0^1 du\; l(z,G,G^z), \tag{2.14}$$

where

$$
\begin{aligned}
l(z,G,G^z) \;=\; & \sum_{i=1}^{\infty}\Big(\frac{-\Delta G}{G}\Big)^i\Big\{-\frac{z^2}{\sinh(z)^2}\frac{4}{G}\frac{1}{(i+1)(i+2)}+\frac{z}{\tanh(z)}\frac{8}{G}\frac{1}{i(i+2)}\Big\}\\
& -\frac{2z}{\tanh(z)}\sum_{i=2}^{\infty}\Big(-\frac{\Delta G}{G}\Big)^i\Big(1+\frac{1}{i}\Big)\frac{1}{G^2}+\frac{4}{G}\frac{z}{\tanh(z)}\Big(\frac{\Delta G}{G^2}+\frac{z}{\tanh(z)}-1\Big)\\
& +4\frac{z}{\tanh(z)}\Big(\frac{z}{\tanh(z)}-1\Big)\,. \qquad (2.15)
\end{aligned}
$$

Here (2.7) was used to obtain the $\frac{1}{G}$ subtraction for the last term. We now expand the trigonometric functions using (1.3) and

$$
\frac{z^2}{\sinh^2(z)}=\sum_{n=0}^{\infty}\frac{\mathcal{B}_{2n}}{(2n)!}(1-2n)(2z)^{2n} \quad . \qquad (2.16)
$$

This leads directly to the following closed-form expression for the expansion coefficients:

$$
\begin{aligned}
a^{(2)}_{\rm spin,main}(n) \;=\; & 2^{2n+4}(2n+1)!\Bigg\{\sum_{i=1}^{n+2}\sum_{M=i}^{n+2}\sum_{\substack{m_1,\ldots,m_i=1\\ \sum m_i=M}}^{M}\sum_{k_1=1}^{m_1}\tilde f(m_1,k_1)\cdots\sum_{k_i=1}^{m_i}\tilde f(m_i,k_i)\\
& \times\beta\Big(\sum_{j=1}^{i}k_j\Big)\Big[\frac{4}{(i+1)(i+2)}\Big(2(n-M)+3\Big)+\frac{8}{i(i+2)}\Big]t_{n+2-M}\\
& -\sum_{i=2}^{n+2}\sum_{M=i}^{n+2}\sum_{\substack{m_1,\ldots,m_i=1\\ \sum m_i=M}}^{M}\sum_{k_1=1}^{m_1}\tilde f(m_1,k_1)\cdots\sum_{k_i=1}^{m_i}\tilde f(m_i,k_i)\,\beta\Big(\sum_{j=1}^{i}k_j-1\Big)\frac{2(i+1)}{i}\,t_{n+2-M}\\
& -4\sum_{m=2}^{n+2}\sum_{k=2}^{m}\tilde f(m,k)\beta(k-1)t_{n+2-m}-8(n+2)t_{n+2}\Bigg\} \qquad (2.17)
\end{aligned}
$$

where we have defined the short-hand:

$$
t_n \;\equiv\; \frac{\mathcal{B}_{2n}}{(2n)!} \quad . \qquad (2.18)
$$

The full two-loop expansion coefficients in (1.7) are now given in closed-form by combining (2.6) and (2.17):

$$
a^{(2)}_{\rm spin}(n) \;=\; a^{(2)}_{\rm spin,main}(n)+a^{(2)}_{{\rm spin},\delta{\rm m}}(n)\;. \qquad (2.19)
$$

3 Expansion coefficients for scalar QED

The case of scalar QED can be treated completely analogously, starting from any of the various representations for $\mathcal{L}^{(2)}_{\rm scal}(B)$ given in [13, 14, 15]. We will give here only the final result for the weak field expansion coefficients (as defined by (1.8)):

$$\begin{aligned}
a^{(2)}_{\rm scal}(n) \;\; = \;\; & -2^{2n+4}(2n+1)!\Biggl\{\sum_{M=2}^{n+2}\sum_{k=2}^{M}\tilde{f}(M,k)\,\beta(k-1)\,s_{n+2-M} \\
&+\sum_{i=2}^{n+2}\sum_{M=i}^{n+2}\sum_{\substack{m_1,\dots,m_i=1\\ \sum m_i=M}}^{M}\sum_{k_1=1}^{m_1}\tilde{f}(m_1,k_1)\cdots\sum_{k_i=1}^{m_i}\tilde{f}(m_i,k_i)\,\beta\Bigl(\sum_{j=1}^{i}k_j-1\Bigr)\,s_{n+2-M} \\
&+\sum_{i=1}^{n+2}\sum_{M=i}^{n+2}\sum_{\substack{m_1,\dots,m_i=1\\ \sum m_i=M}}^{M}\sum_{k_1=1}^{m_1}\tilde{f}(m_1,k_1)\cdots\sum_{k_i=1}^{m_i}\tilde{f}(m_i,k_i)\,\beta\Bigl(\sum_{j=1}^{i}k_j\Bigr) \\
&\qquad\qquad\times\frac{4}{i+2}\biggl[\frac{1}{i+1}-(2n-2M+3)\biggr]\,s_{n+2-M} \\
&+\sum_{i=1}^{n+1}\sum_{M=i}^{n+1}\sum_{\substack{m_1,\dots,m_i=1\\ \sum m_i=M}}^{M}\sum_{k_1=1}^{m_1}\tilde{f}(m_1,k_1)\cdots\sum_{k_i=1}^{m_i}\tilde{f}(m_i,k_i) \\
&\qquad\qquad\times\frac{2}{(i+1)(i+2)}\,\beta\Bigl(\sum_{j=1}^{i}k_j+1\Bigr)\,s_{n-M+1} \\
&-s_{n+1}+2s_{n+2}\Bigl[2+(2n+2)\Bigl(4-3\psi(2n+3)-3\gamma\Bigr)\Bigr]\Biggr\}.
\end{aligned}\tag{3.1}$$

Here we have defined the short-hand notation:

$$s_n \equiv -\Bigl(1-2^{1-2n}\Bigr)\frac{\mathcal{B}_{2n}}{(2n)!}\quad . \tag{3.2}$$

4 Conclusions

We have found closed-form expressions for the weak field expansion coefficients of the two loop corrections to the renormalized Euler-Heisenberg Lagrangians in a purely magnetic (or purely electric) field. As a check, we have verified that eqs.(2.19) and (3.1) indeed reproduce the known low order coefficients in these expansions [14, 8]:

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\rm spin}[B] &= \Big(\frac{\alpha}{4\pi}\Big)^2\frac{B^2}{81}\Bigg[64\Big(\frac{eB}{m^2}\Big)^2-\frac{1219}{25}\Big(\frac{eB}{m^2}\Big)^4+\frac{135308}{1225}\Big(\frac{eB}{m^2}\Big)^6-\dots\Bigg],\\
\mathcal{L}^{(2)}_{\rm scal}[B] &= \Big(\frac{\alpha}{4\pi}\Big)^2\frac{B^2}{81}\Bigg[\frac{275}{8}\Big(\frac{eB}{m^2}\Big)^2-\frac{5159}{200}\Big(\frac{eB}{m^2}\Big)^4+\frac{2255019}{39200}\Big(\frac{eB}{m^2}\Big)^6-\dots\Bigg].
\end{aligned}
\tag{4.1}
$$

Although the closed-form formulae (2.19) and (3.1) are significantly more complicated than the corresponding ones for the case of a self-dual field [18, 19], their structure is still similar insofar as they can, using (2.9), be written in terms of folded sums of Bernoulli numbers with factorial coefficients. Of course, it is quite possible that these formulas can still be simplified.

We note that the two-loop expansion coefficients are still rational numbers, after extracting a factor of $(\alpha/\pi)^2$, just as the one-loop coefficients are rational after extracting a factor of (α/π). A question of obvious interest is whether this property persists to higher loop orders. Based on a comparison with what is known about the coefficients of the QED β - functions [21, 22, 23, 24, 25] we consider it likely that rationality will be found to hold at least for the quenched (order N_f) contributions to the Euler-Heisenberg Lagrangians at arbitrary loop order.

Finally, since in the worldline formalism the magnetic Green's function G^z is the basic ingredient appearing in the integral representations for all processes involving constant magnetic fields [13, 15], we expect the Faulhaber expansion (2.7) to become useful for other calculations of this type, including possibly higher loop orders.

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